Journal of Approximation Theory **105**, 252–262 (2000)
doi:10.1006/jath.2000.3467, available online at http://www.idealibrary.com on IDEAL®

Geometry and Topology of Continuous Best and Near Best Approximations

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Communicated by András Kroó

Received March 15, 1999; accepted in revised form February 25, 2000

The existence of a continuous best approximation or of near best approximations of a strictly convex space by a subset is shown to imply uniqueness of the best approximation under various assumptions on the approximating subset. For more general spaces, when continuous best or near best approximations exist, the set of best approximants to any given element is shown to satisfy connectivity and radius constraints. © 2000 Academic Press

Key Words: best approximation; near best approximation; continuous selection; Chebyshev set; strictly convex space; uniformly convex space; contractible set; tangent hypercone; modulus of convexity; Chebyshev radius.

1. INTRODUCTION

An approximant within ε of the best possible is usually satisfactory from a practical standpoint. The idea of extending best approximation to near best approximation has been investigated for many years; see [1; 3; 8, p. 162]. "Near best" has several possible interpretations, and the one used here is a map ϕ: $X \to M$ for which $\|x - \phi(x)\| \leqslant \|x - M\| + \varepsilon$. In this paper we investigate when best or near best approximations of a normed linear space by elements of a subset can have a continuous selection. Under

0021-9045/00 $35.00

various conditions on the ambient space and the subset we derive topological and geometric consequences of the existence of continuous best and near best approximations.

The paper is organized as follows. Section 2 gives our main theorems. We show that continuity of best approximation implies uniqueness when the ambient space is strictly convex. We also show that continuity of near best approximations with arbitrarily small ε is enough to guarantee uniqueness in a strictly convex space when the subset is boundedly compact and closed. The next two sections generalize these results by removing the condition of strict convexity on the ambient space. In Section 3 the set of best approximants to a point may no longer be a singleton but is shown to have topological properties such as contractibility, while Section 4 establishes an upper bound on the Chebyshev radius of the approximant set in terms of the modulus of convexity.

2. CONTINUITY CONDITIONS FOR UNIQUE BEST APPROXIMATION

In this section we demonstrate that the existence of a continuous best approximation or of a suitable family of continuous near best approximations defined on a strictly convex space X and taking values in a suitable subset M necessarily implies that M has the unique best approximation property.

Let X be a normed linear space, always taken to be over the reals. The space X is *strictly convex* iff whenever x and y are distinct unit vectors all nontrivial convex combinations of the two have norm less than 1. For x in X and $r \geqslant 0$, let $B(x, r)$ denote the closed ball centered on x of radius r, with $\partial B(x, r)$ its boundary sphere. For any subset A we write $cl(A)$ for its closure.

If M is a subset of X, we denote by $P_M(x)$ the set $\{m \in M: \|x - m\| = \|x - M\|\}$. An element of $P_M(x)$ is called a *best approximation* to x; P_M is a set-valued function which associates to each x in X the (possibly empty) set of all its best approximations. The terminology *metric projection operator* is also used for P_M (see [8]).

If $P_M(x)$ is nonempty for each x in X, M is said to be *proximinal*. If $P_M(x)$ is a singleton for each x in X, M is called a *Chebyshev* set. In the latter case, we use a lower-case "p" to denote the metric projection function; that is, when M is Chebyshev, $p_M: X \to M$ is the unique function satisfying $P_M(x) = \{p_M(x)\}$ for all x in X.

A *selection* for a set-valued function Φ is a function ϕ such that $\phi(x)$ is in $\Phi(x)$ for each x. Given a nonempty subset A of X, a *best approximation* of A by M is a function $\phi: A \to M$ such that $\|x - \phi(x)\| = \|x - M\|$ for all

x in A. Thus a best approximation is a selection for the metric projection operator. Our first result, a version of which appeared in [5], deals with the case when continuous selections exist. See also [7].

THEOREM 2.1. *Let X be a strictly convex normed linear space, and let M be a subset of X. Let ϕ: $X \to M$ be a continuous best approximation of X by M. Then M is a Chebyshev set.*

Proof. Since $\phi(x) \in P_M(x)$ for all x, $P_M(x)$ is nonempty. Given x in X, let m belong to $P_M(x)$. For y in the line segment $[m, x)$, and u in $P_M(y)$, $\|u-x\| \leqslant \|u-y\| + \|y-x\| \leqslant \|m-y\| + \|y-x\| = \|m-x\| \leqslant \|u-x\|$. Hence, the inequalities are all equalities, u is in $P_M(x)$, $P_M(y) \subseteq P_M(x)$, and $m \in P_M(y)$. Since $\|u-x\| = \|u-y\| + \|y-x\|$, a consequence of strict convexity is that u, y, and x are collinear. So $u = m$ and $P_M(y) = \{m\}$. Since ϕ is directionally continuous at x and $\phi([m, x)) = \{m\}$, it follows that $\phi(x) = m$. Thus $P_M(x) = \{\phi(x)\}$ is a singleton set. ∎

In case X is not strictly convex, there do exist subsets M of X for which a continuous best approximation of X by M exists without M being Chebyshev.

EXAMPLE 2.2. Let $X = \mathscr{R}^2$ equipped with the l_1-norm $\|(a, b)\| = |a| + |b|$. With this norm, $\mathscr{R}^2$ is not strictly convex. Let $M = \{(a, b): b = \pm a\}$. Then, with $x = (0, 1)$, $\|x - M\| = \inf\{|a| + |\pm a - 1| : a \in \mathscr{R}\} = 1$, and $P_M(x) = \{(a, |a|) : |a| \leqslant 1\}$. So M is not a Chebyshev set. However, a continuous best approximation of $\mathscr{R}^2$ by M does exist, namely, the map ϕ given by: $\phi(a, b) = \min\{a, b\}$ for (a, b) in the first quadrant, with similar prescriptions in the other three quadrants. Note, in addition, that M is *almost convex* (see Huotari and Li [4]); i.e., any closed ball that does not meet M lies inside arbitrarily large closed balls that also do not meet M.

For a subset A of X and a positive number ε, an *ε-near best approximation* of A by M is a map ϕ: $A \to M$ such that $\|x - \phi(x)\| \leqslant \|x - M\| + \varepsilon$ for all x in A (see [1, 8]). A subset M of X is *boundedly compact* iff the closure of $M \cap B$ is compact for each closed ball B in X.

THEOREM 2.3. *Let X be a strictly convex normed linear space, and let M be a closed, boundedly compact subset of X. Suppose that for each $\varepsilon > 0$ there exists a continuous ε-near best approximation ϕ: $X \to M$ of X by M. Then M is a Chebyshev set.*

Proof. Since M is boundedly compact and closed, any sequence $\{m_n\}$ in M with $\lim_{n\to\infty} \|x - m_n\| = \|x - M\|$ accumulates at a point m in M, and so $P_M(x)$ is nonempty. Thus M is proximinal.

Let x_0 be a point in X with $r = \|x_0 - M\| > 0$. Given an integer $n \geqslant 1$, let $\phi_n\colon X \to M$ be continuous with $\|x - \phi_n(x)\| \leqslant \|x - M\| + \frac{1}{n}$ for all x in X. Then $\phi_n\colon B(x_0, r) \to M$ and $\|\phi_n(x) - x_0\| \geqslant r$ for x in $B(x_0, r)$. Let

$$\pi\colon \{x\colon \|x - x_0\| \geqslant r\} \to \{x\colon \|x - x_0\| = r\} = \partial B(x_0, r)$$

be the radial retraction, i.e.,

$$\pi(x) = x_0 + r\,\frac{x - x_0}{\|x - x_0\|}.$$

Then $\pi \circ \phi_n\colon B(x_0, r) \to \partial B(x_0, r)$. Now $\phi_n(x)$, for x in $B(x_0, r)$, satisfies $\|\phi_n(x) - x_0\| \leqslant \|x - M\| + \frac{1}{n} + \|x - x_0\| \leqslant 2\,\|x - x_0\| + \|x_0 - M\| + \frac{1}{n} \leqslant 3r + 1$. Hence, $\phi_n(B(x_0, r)) \subseteq M \cap B(x_0, 3r + 1)$ and $\phi_n(B(x_0, r))$ is a bounded subset of M. So $cl(\phi_n(B(x_0, r)))$ is compact since M is boundedly compact. Let $\rho\colon X \to X$ be the reflection through x_0, i.e., $\rho(y) = x_0 + (x_0 - y)$. Then $cl(\rho \circ \pi \circ \phi_n(B(x_0, r))) = \rho \circ \pi(cl\phi_n(B(x_0, r)))$ is a compact subset of $\partial B(x_0, r)$, and $\rho \circ \pi \circ \phi_n$ is a continuous function from $B(x_0, r)$ into this set.

Rothe's Theorem (see [9, p. 27]), a version of Schauder's Theorem, asserts that any continuous map from the closed ball B into X taking ∂B into a compact subset of B has a fixed point. Hence, for each n, $\rho \circ \pi \circ \phi_n$ has a fixed point x_n in $B(x_0, r)$. Thus,

$$x_n = \rho \circ \pi \circ \phi_n(x_n) = 2x_0 - \pi \circ \phi_n(x_n),$$

and

$$\pi \circ \phi_n(x_n) = 2x_0 - x_n.$$

It follows that the points x_n, x_0, $2x_0 - x_n = \pi \circ \phi_n(x_n)$, and $\phi_n(x_n)$ are consecutive collinear points (with the last two possibly equal), and thus $\|\phi_n(x_n) - x_n\| \geqslant \|\pi \circ \phi_n(x_n) - x_n\| = \|2x_0 - 2x_n\| = 2r$. In addition, for each point m in M,

$$\|x_n - m\| \geqslant \|x_n - \phi_n(x_n)\| - \frac{1}{n} \geqslant 2r - \frac{1}{n}. \tag{1}$$

Again because M is boundedly compact, the sequence $\{\phi_n(x_n)\}$ in $M \cap B(x_0, 3r + 1)$ has a convergent subsequence with limit u in X. Then the sequence $\{x_n\}$, where $x_n = \rho \circ \pi \circ \phi_n(x_n)$, has a convergent subsequence with limit $\rho \circ \pi(u) = x_\infty \in \partial B(x_0, r)$.

Moreover, for each m in M, because of (1)

$$\|(x_\infty - x_0) + (x_0 - m)\| = \|x_\infty - m\| \geqslant 2r.$$

If m is in $P_M(x_0)$, then $\|x_0 - m\| = r$. By strict convexity, used here for the first time, when $\|(a+b)/2\| \geqslant r$ and $\|a\| = \|b\| = r$, then $a = b$. So we conclude that $x_\infty - x_0 = x_0 - m$ and $m = 2x_0 - x_\infty$. Thus $P_M(x_0) = \{2x_0 - x_\infty\}$ is a singleton set. This being true for all x_0, M is Chebyshev. ∎

Since metric projection to a closed, boundedly compact Chebyshev subset is continuous (see [8, p. 390]), our result says that in a strictly convex space the existence of continuous arbitrarily precise near best approximations is equivalent to the existence of a unique best approximation which is continuous.

A set M is *positively homogeneous* provided $\lambda M = M$ for each positive number λ. If M is nonempty, closed and positively homogeneous, then 0 is in M.

The next theorem gives conditions under which the existence of a single continuous near-best approximation is sufficient to guarantee continuous unique best approximation.

COROLLARY 2.4. *Let X be a strictly convex normed linear space, and let M be a closed, boundedly compact, positively homogeneous subset of X. Suppose that for some $\varepsilon > 0$ there exists a continuous ε-near best approximation ϕ: $X \to M$ of X by M. Then M is a Chebyshev set.*

Proof. For $\lambda > 0$, consider the maps ϕ_λ defined by $\phi_\lambda(x) = \lambda\phi(\frac{x}{\lambda})$ for x in X. The map ϕ_λ is continuous. It is a $\lambda\varepsilon$-near best approximation of X by M since $\|\phi_\lambda(x) - x\| = \lambda\, \|\phi(\frac{x}{\lambda}) - \frac{x}{\lambda}\| \leqslant \lambda(\|\frac{x}{\lambda} - M\| + \varepsilon) = \|x - M\| + \lambda\varepsilon$ for x in X. Theorem 2.3 can be applied to this family of maps. ∎

A special case of some interest is when M consists of a finite union of finite-dimensional subspaces. See [5, Theorem 3.6].

3. CONTINUITY CONDITIONS FOR CONNECTIVITY OF $P_M(x)$

Theorems 2.1 and 2.3 are special cases of a family of results that we now develop in more detail. The following result was established in the proof of Theorem 2.1 without the requirement that X be strictly convex.

LEMMA 3.1. *Let X be a normed linear space, M a subset of X, x an element of X, and m an element of $P_M(x)$. Then for each $y \in [m, x]$, $\{m\} \subseteq P_M(y) \subseteq P_M(x)$.*

For x in X and S a subset of X, let $x * S$ denote the union of all line segments with one endpoint in S and the other equal to x. The *tangent hypercone* to the closed ball B at the point $x \in \partial B$ is the union of all closed affine hyperplanes containing x but not meeting the interior of B. Such hyperplanes are said to be tangent to B at x. By Mazur's version of the Hahn–Banach Theorem (see [2, p. 23]) any line (or flat) through x that does not meet the interior of B lies in a closed affine hyperplane tangent to B at x, and hence is in the tangent hypercone to B at x.

A subset A of a space X is called *contractible* to a point a_0 in A if there is a continuous map $h: A \times [0, 1] \to A$ with $h(a, 0) = a$ and $h(a, 1) = a_0$ for all a in A (see [10, p. 25]).

THEOREM 3.2. *Let X be a normed linear space and M a subset of X. Let x be an element of X with $r = \|x - M\| > 0$ and $P_M(x)$ nonempty. Let $\phi: x * P_M(x) \to M$ be a continuous best approximation of $x * P_M(x)$ by M. Then*

(i) *$P_M(x)$ is contractible to $\phi(x)$;*

(ii) *$P_M(x)$ is a subset of the tangent hypercone to $B(x, r)$ at $\phi(x)$.*

Proof. Define $h: P_M(x) \times [0, 1] \to P_M(x)$ by $h(m, t) = \phi((1-t)\,m + tx)$. By Lemma 3.1 the range of this map is a subset of $P_M(x)$. Since $P_M(m) = \{m\}$ for $m \in M$, $\phi(m) = m$ for points $m \in P_M(x)$, h is a homotopy between the identity map on $P_M(x)$ and the constant map with output $\phi(x)$.

If $P_M(x)$ consists of two or more points, let one of them be $\phi(x)$ and let another be m. Let y be a point on the open interval (m, x) with $\phi(\bar{y}) \neq m$ for $\bar{y}$ in $[x, y)$. Such a point y exists by continuity of ϕ at x. Then as in Theorem 2.1 $\|\phi(x) - x\| = \|\phi(y) - x\| = \|\phi(y) - y\| + \|y - x\|$. Choose y' on the open interval $(x, \phi(y))$ such that $\|y' - x\| = \|y - x\|$. Then all points on the line segment $[y, y']$ are equidistant from x. This follows from the fact that each such point has distance to x bounded above by $\|y' - x\| = \|y - x\|$, and distance to $\phi(y)$ bounded above by $\|y' - \phi(y)\| = \|x - \phi(y)\| - \|x - y'\| = \|x - \phi(y)\| - \|x - y\| = \|x - \phi(x)\| - \|x - y\| = \|y - \phi(x)\| = \|y - \phi(y)\|$. Since the sum of these two distances is thus bounded above and below by $\|x - \phi(y)\|$, all bounds are equalities. Expanding this line segment radially from x by a factor of $\frac{\|m - x\|}{\|y - x\|}$, we obtain the line segment $[m, \phi(y)]$ and each point on this line segment is equidistant from x as well. Varying y toward x, we find that the points on the line segment $[m, \phi(x)]$ are all at distance $r = \|x - M\|$ from x. Thus the line through m and $\phi(x)$ is a tangent line to $B(x, r)$ at $\phi(x)$ and hence m lies in the tangent hypercone to $B(x, r)$ at $\phi(x)$. ∎

Since $P_M(x)$ is contractible, it is path-connected and has trivial homology.

When a continuous best approximation is defined on all of X, *a fortiori* $P_M(x)$ is contractible for every x in X. Thus M belongs to a class of sets called "C_2" by Klee [6] (see also [8, p. 370]). If, in addition, X is strictly convex, each tangent hypercone intersects the ball at a single point, and $P_M(x) = \{\phi(x)\}$ for all x. This implies Theorem 2.1.

Note that the conclusions of Theorem 3.2 are illustrated in Example 2.2 by the subset $P_M(0, 1)$.

Versions of Theorem 2.3 also hold without the assumption of strict convexity.

A set M is *approximatively compact* iff whenever $x \in X$ and $\{m_n\}$ is a sequence in M such that $\lim_{n \to \infty} \|x - m_n\| = \|x - M\|$, then $\{m_n\}$ has a convergent subsequence with limit in M. An approximatively compact set is always closed. Conversely, a set that is both closed and boundedly compact is approximatively compact since the sequence $\{m_n\}$ is bounded and hence has a convergent subsequence.

We remark that if M is an approximatively compact set in a normed linear space, then $P_M(x)$ is compact for each x in X. Indeed, any sequence $\{m_n\}$ in $P_M(x)$ is a sequence in M with $\|m_n - x\| = \|M - x\|$, and by the definition of approximative compactness has a convergent subsequence with limit in M, and hence in $P_M(x)$.

If M is an approximatively compact Chebyshev set, then p_M is continuous on X (see [8, p. 390]). Theorem 2.1 says that if P_M has a continuous selection in a strictly convex space X, then M is Chebyshev. In the absence of strict convexity, Example 2.2 above shows that even if M is approximatively compact and P_M has a continuous selection, M need not be Chebyshev.

The next result says that if we vary the hypotheses of Theorem 3.2 by requiring near best approximations but insist that M be approximatively compact, then $P_M(x)$ retains the basic topological property of being connected.

PROPOSITION 3.3. *Let M be approximatively compact in the normed linear space X, and let x be an element of X. Suppose that for each $\varepsilon > 0$ there is a continuous ε-near best approximation $\phi_\varepsilon: x * P_M(x) \to M$ of $x * P_M(x)$ by M. Then $P_M(x)$ is connected.*

Proof. If, to the contrary, $P_M(x)$ were not connected, there would exist distinct open sets U_1 and U_2 covering $P_M(x)$ with $P_M(x) \cap U_1 \neq \emptyset \neq P_M(x) \cap U_2$ and $P_M(x) \cap U_1 \cap U_2 = \emptyset$. Let $A = P_M(x) \cap U_1^c$ and $B = P_M(x) \cap U_2^c$ where S^c denotes the complement of the set S in X. Then, by our remark above, A and B are compact. They are also disjoint, nonempty and their union contains $P_M(x)$. Since the distance from A to B is positive, there exist disjoint open neighborhoods V_1 of A and V_2 of B.

Pick $z_1 \in A$ and $z_2 \in B$. Choose ε sufficiently small so that $B(z_i, \varepsilon) \subseteq V_i$, $i = 1, 2$. The ε-near best approximation ϕ_ε takes $[z_1, x] \cup [x, z_2]$ into a path in M from $\phi_\varepsilon(z_1)$ to $\phi_\varepsilon(z_2)$ so that $\|\phi_\varepsilon(z_i)-z_i\| \leqslant \|z_i - M\| + \varepsilon = \varepsilon$ for $i=1, 2$. Then $\phi_\varepsilon(z_i) \in V_i$ for $i=1, 2$.

Because $V_1 \cap V_2 = \emptyset$ there is a point m_ε in $(V_1 \cup V_2)^c$ with $m_\varepsilon = \phi_\varepsilon(x_\varepsilon)$ for some x_ε in $[z_1, x] \cup [x, z_2]$, and we claim that $\|m_\varepsilon - x\| \leqslant \|M - x\| + \varepsilon$. Indeed, $\|m_\varepsilon - x\| \leqslant \|m_\varepsilon - x_\varepsilon\| + \|x_\varepsilon - x\| \leqslant \|M - x_\varepsilon\| + \varepsilon + \|x_\varepsilon - x\| \leqslant \|z_i - x_\varepsilon\| + \|x_\varepsilon - x\| + \varepsilon = \|z_i - x\| + \varepsilon = \|M - x\| + \varepsilon$ where $i=1$ or 2 according as x_ε is in $[z_1, x]$ or $[z_2, x]$.

As ε approaches 0, $\|m_\varepsilon - x\|$ approaches $\|x - M\|$. Since m_ε is in M for each $\varepsilon > 0$ and M is approximatively compact, we can find a sequence in the set $\{m_\varepsilon : \varepsilon > 0\}$ with limit m in M satisfying $\|m - x\| = \|M - x\|$. So m is in $P_M(x) = A \cup B \subseteq V_1 \cup V_2$. But m is the limit of points in the complement of the open set $V_1 \cup V_2$, for a contradiction. ∎

Connectivity of $P_M(x)$ implies that if $P_M(x)$ contains more than one point, it contains an uncountable number of points.

COROLLARY 3.4. *Let X be a normed linear space, and M an approximatively compact subset of X which is countably proximinal (i.e., $P_M(x)$ is nonempty and countable) for each x in X. Suppose that for each $\varepsilon > 0$ there exists a continuous ε-near best approximation $\phi: X \to M$ of X by M. Then M is a Chebyshev set.*

Proof. By Proposition 3.3, for each x, $P_M(x)$ is connected. The only nonempty countable connected set is a singleton. ∎

Note that if M is proximinal and a countable union of Chebyshev sets, then M is countably proximinal.

THEOREM 3.5. *Let X be a normed linear space, M a closed, boundedly compact subset of X, and x an element of X with $r = \|x - M\| > 0$. If, for each $\varepsilon > 0$, there exists a continuous ε-near best approximation $\phi: B(x, r) \to M$ of $B(x, r)$ by M, then*

(i) *$P_M(x)$ is connected, and*

(ii) *$P_M(x)$ is a subset of the tangent hypercone to $B(x, r)$ at some point on $\partial B(x, r)$.*

Proof. Since a closed, boundedly compact subset is approximatively compact, connectedness of $P_M(x)$ is a consequence of Proposition 3.3.

The remainder of the proof repeats the proof of Theorem 2.3 (with x_0 in place of x) until strict convexity is mentioned. For each point m in $P_M(x_0)$, we have that $\|x_\infty - m\| \geqslant 2r$ where $r = \|m - x_0\| = \|x_\infty - x_0\| = \|M - x_0\|$. Since $\|\lambda(x_0 - x_\infty) + (1-\lambda)(m - x_0)\| \leqslant r$ for $0 \leqslant \lambda \leqslant 1$ and $\|\frac{1}{2}(x_0 - x_\infty) + \frac{1}{2}(m - x_0)\| \geqslant r$, it follows that the line joining $2x_0 - x_\infty$ to m does not meet the interior of $B(x_0, r)$. Hence m is in the tangent hypercone to $B(x_0, r)$ at $2x_0 - x_\infty$. ∎

4. BOUNDS ON THE RADIUS OF $P_M(x)$

In addition to topological properties, one can also deduce constraints on the radius of the approximant set $P_M(x)$. Recall from [2, p. 111] that the *Chebyshev radius* of a set A with respect to a point x is the number $\sup\{\|x-a\| : a \in A\}$. The following lemma restates a result obtained in the first part of the proof of Theorem 2.3 (up through Eq. (1)).

LEMMA 4.1. *Let X be a normed linear space, M a boundedly compact subset of X, and x an element of X with $r = \|x - M\| > 0$. Suppose that for some ε, with $0 < \varepsilon < 2r$, there exists a continuous ε-near best approximation ϕ: $B(x, r) \to M$ of $B(x, r)$ by M. Then there exists a point $\bar{x}$ in $\partial B(x, r)$ such that $\|\bar{x} - m\| \geqslant 2r - \varepsilon$ for all m in M.*

In particular, any point m in $P_M(x)$ is at distance $2r - \varepsilon$ or more from $\bar{x}$ and at distance r from x. In a Hilbert space the right triangle with legs from m to $\bar{x}$ and from m to $2x - \bar{x}$ has hypotenuse $2r$ and one leg of length $\geqslant 2r - \varepsilon$. Accordingly its other leg, between m and $2x - \bar{x}$, has length at most $\sqrt{4\varepsilon r - \varepsilon^2}$. Thus the Chebyshev radius of $P_M(x)$ with respect to the point $2x - \bar{x}$ is less than or equal to $\sqrt{4\varepsilon r - \varepsilon^2}$.

For more general spaces we consider two notions associated with the geometry of the unit sphere.

In a normed space X define a function δ_X called the *modulus of convexity* of X by $\delta_X(t) = \inf\{1 - \frac{1}{2}(\|x + y\|) : \|x\| = \|y\| = 1, \|x - y\| \geqslant t\}$ for t a real number. The function δ_X is a nondecreasing function and satisfies $\delta_X(0) = 0$ (see, e.g., [2, p. 145]), $\delta_X(2) \leqslant 1, \delta_X(t) = \infty$ for $t > 2$. If the modulus is strictly positive-valued for t strictly positive, the space X is said to be *uniformly convex*.

A related measure of convexity is the function ω_X. For $t < 1$, let $\omega_X(t) = \sup\{\|x - y\| : 1 - t \leqslant \|\frac{x+y}{2}\|, \|x\| = \|y\| = 1\}$, and set $\omega_X(t) = 2$ for $t \geqslant 1$. Then ω_X is a nondecreasing function. In a uniformly convex space X, $\omega_X(t) > 0$ for $t > 0$ and $\lim_{t \to 0} \omega_X(t) = 0$.

The following inequality can be easily verified,

$$\omega_X(t) \leqslant \sup \{s: \delta_X(s) \leqslant t\}$$

for all real numbers t. Equality occurs if $t < 0$ or $\geqslant 1$, or if X is finite-dimensional.

THEOREM 4.2. *Let X be a normed linear space, M a boundedly compact subset of X, and x an element of X with $\|x-M\|=r>0$. Suppose that for some ε, with $0<\varepsilon<2r$, there exists a continuous ε-near best approximation ϕ: $B(x,r)\to M$ of $B(x,r)$ by M. Then there exists a point x' on $\partial B(x,r)$ such that $R_{x'}(P_M(x))$, the Chebyshev radius of $P_M(x)$ with respect to x', satisfies*

$$R_{x'}(P_M(x))\leqslant r\omega_X\left(\frac{\varepsilon}{2r}\right).$$

Proof. Let $x'=2x-\bar{x}$ where $\bar{x}$ is as in Lemma 4.1. Then $u=(m-x)/r$ and $v=(x'-x)/r$ are unit vectors in X with $\|(u+v)/2\|=\|(m-\bar{x})/2r\|\geqslant 1-\varepsilon/2r$. Hence, $\|(m-x')/r\|=\|u-v\|\leqslant\omega_X(\varepsilon/2r)$. ∎

If X is uniformly convex and there is a continuous ε-near best approximation of X by the closed set M for every $\varepsilon>0$, the diameter of $P_M(x)$ is zero for each x and so M is Chebyshev. This conclusion also follows from Theorem 2.3 since uniform convexity implies strict convexity.

Example 2.2 illustrates that in some non-strictly convex spaces the lower bound of $2r-\varepsilon$ does not limit the radius of the approximating set. With the norm as in 2.2, the points (0,1), (1,0), and $(-1/2,\ -1/2)$ are unit vectors and their distances apart are all equal to 2. The fact that any one of them is "far" from the other two does not force those two to be close together. Indeed, in this case $\omega_x(t)\equiv 2$ for $t\geqslant 0$.

ACKNOWLEDGMENTS

V. Kůrková was partially supported by GA AV Grant A2030602, GA ČR Grant 201/99/0092, and GA ČR Grant 201/00/1489. Collaboration of V. Kůrková and A. Vogt was supported by an NRC COBASE grant.

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